International Journal of Mechanical and Industrial Technology ISSN 2348-7593 (Online)
Vol. 11, Issue 1, pp: (18-22), Month: April 2023 - September 2023, Available at: www.researchpublish.com

A Fractional Definite Integral Formula and Its Applications

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DOI: https://doi.org/10.5281/zenodo.8154710

Published Date: 17-July-2023

Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we obtain a fractional definite integral formula. Moreover, some examples are provided to illustrate the applications of this formula. In fact, our formula is a generalization of traditional calculus formula.**

Keywords: **Jumarie type of R-L fractional calculus, new multiplication, fractional analytic functions, fractional definite integral formula.**

I. INTRODUCTION

In the second half of the 20th century, a considerable number of studies on fractional calculus were published in the engineering literature. In fact, fractional calculus has many applications in physics, mechanics, biology, electrical engineering, viscoelasticity, control theory, economics, and other fields [1-14]. There is no doubt that fractional calculus has become an exciting new mathematical method to solve different problems in mathematics, science, and engineering. However, the definition of fractional derivative is not unique. There are many useful definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov fractional derivative, Jumarie's modified R-L fractional derivative [15-19]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we study a fractional definite integral formula. In addition, we give some examples to illustrate the applications of this formula. In fact, our formula is a generalization of ordinary calculus formula.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1 ([20])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([21])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

Definition 2.3 ([22])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0 < \alpha \le 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([23])**:** If $0 < \alpha \le 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}. \tag{6}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{8}$$

Definition 2.5 ([24])**:** If $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{9}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(g_\alpha(x^\alpha)\right)^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha n}. \tag{12}$$

Definition 2.6 ([25])**:** Let $0 < \alpha \le 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} x^\alpha. \tag{13}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.7 ([26])**:** If $0 < \alpha \le 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{14}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{15}$$

and

$$sin_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{16}$$

III. RESULTS AND EXAMPLES

In this section, we introduce a fractional definite integral formula and provide some examples to illustrate its applications. At first, we need a lemma.

Lemma 3.1: *If* $0<\alpha\leq 1,(-1)^\alpha=-1$, r *is a real number, and* $f_\alpha(x^\alpha)$ *is a* α*-fractional analytic function on* $[-r,r]$, *then*

$$\left({}_{-r}I_r^\alpha\right)[f_\alpha(x^\alpha)]=\left({}_0I_r^\alpha\right)[f_\alpha(x^\alpha)+f_\alpha(-x^\alpha)]. \tag{17}$$

Proof Since $\left({}_0I_r^\alpha\right)[f_\alpha(-x^\alpha)]$

$$\begin{aligned}
&=\left({}_0I_r^\alpha\right)\left[f_\alpha(-x^\alpha)\otimes_\alpha\left({}_0D_r^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\right]\\
&=-\left({}_0I_r^\alpha\right)\left[f_\alpha(-x^\alpha)\otimes_\alpha\left({}_0D_r^\alpha\right)\left[-\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\right]\\
&=\left({}_rI_0^\alpha\right)\left[f_\alpha(-x^\alpha)\otimes_\alpha\left({}_0D_r^\alpha\right)\left[-\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\right]\\
&=\left({}_{-r}I_0^\alpha\right)\left[f_\alpha(x^\alpha)\otimes_\alpha\left({}_0D_r^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\right]\\
&=\left({}_{-r}I_0^\alpha\right)[f_\alpha(x^\alpha)]\,.
\end{aligned} \tag{18}$$

It follows that

$$\begin{aligned}
&\left({}_{-r}I_r^\alpha\right)[f_\alpha(x^\alpha)]\\
&=\left({}_{-r}I_0^\alpha\right)[f_\alpha(x^\alpha)]+\left({}_0I_r^\alpha\right)[f_\alpha(x^\alpha)]\\
&=\left({}_0I_r^\alpha\right)[f_\alpha(-x^\alpha)]+\left({}_0I_r^\alpha\right)[f_\alpha(x^\alpha)]\\
&=\left({}_0I_r^\alpha\right)[f_\alpha(x^\alpha)+f_\alpha(-x^\alpha)].
\end{aligned}$$

Q.e.d.

Theorem 3.2: *Let* $0<\alpha\leq 1$, $(-1)^\alpha=-1$, r *be a real number, and* $f_\alpha(x^\alpha)$ *be an even* α*-fractional analytic function on* $[-r,r]$. *Then the* α*-fractional definite integral*

$$\left({}_{-r}I_r^\alpha\right)\left[f_\alpha(x^\alpha)\otimes_\alpha\left(1+E_\alpha(x^\alpha)\right)^{\otimes_\alpha(-1)}\right]=\left({}_0I_r^\alpha\right)[f_\alpha(x^\alpha)]. \tag{19}$$

Proof By Lemma 3.1 and $f_\alpha(x^\alpha)$ is an even α-fractional analytic function, we obtain

$$\begin{aligned}
&\left({}_{-r}I_r^\alpha\right)\left[f_\alpha(x^\alpha)\otimes_\alpha\left(1+E_\alpha(x^\alpha)\right)^{\otimes_\alpha(-1)}\right]\\
&=\left({}_0I_r^\alpha\right)\left[f_\alpha(x^\alpha)\otimes_\alpha\left(1+E_\alpha(x^\alpha)\right)^{\otimes_\alpha(-1)}+f_\alpha(-x^\alpha)\otimes_\alpha\left(1+E_\alpha(-x^\alpha)\right)^{\otimes_\alpha(-1)}\right]\\
&=\left({}_0I_r^\alpha\right)\left[f_\alpha(x^\alpha)\otimes_\alpha\left(1+E_\alpha(x^\alpha)\right)^{\otimes_\alpha(-1)}+f_\alpha(x^\alpha)\otimes_\alpha\left(1+E_\alpha(-x^\alpha)\right)^{\otimes_\alpha(-1)}\right]
\end{aligned}$$

$$= \left({}_0I_r^{\alpha}\right)\left[f_\alpha(x^\alpha)\otimes_\alpha\left[\left(1+E_\alpha(x^\alpha)\right)^{\otimes_\alpha(-1)}+\left(1+E_\alpha(-x^\alpha)\right)^{\otimes_\alpha(-1)}\right]\right]$$

$$= \left({}_0I_r^{\alpha}\right)\left[f_\alpha(x^\alpha)\otimes_\alpha\left[\left(1+E_\alpha(x^\alpha)\right)^{\otimes_\alpha(-1)}+E_\alpha(x^\alpha)\otimes_\alpha\left(1+E_\alpha(x^\alpha)\right)^{\otimes_\alpha(-1)}\right]\right]$$

$$= \left({}_0I_r^{\alpha}\right)[f_\alpha(x^\alpha)]\,. \qquad \text{Q.e.d.}$$

Example 3.3: If $0<\alpha\leq 1$, $(-1)^\alpha=-1$, then by Theorem 3.2, we have

$$\left({}_{-3}I_3^{\alpha}\right)\left[cos_\alpha(x^\alpha)\otimes_\alpha\left(1+E_\alpha(x^\alpha)\right)^{\otimes_\alpha(-1)}\right]=\left({}_0I_3^{\alpha}\right)[cos_\alpha(x^\alpha)]=sin_\alpha\left(\frac{1}{\Gamma(\alpha+1)}\cdot 3^\alpha\right). \tag{20}$$

And

$$\left({}_{-2}I_2^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 4}\otimes_\alpha\left(1+E_\alpha(x^\alpha)\right)^{\otimes_\alpha(-1)}\right]=\left({}_0I_2^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 4}\right]=\frac{1}{5}\left(\frac{1}{\Gamma(\alpha+1)}\cdot 2^\alpha\right)^{\otimes_\alpha 5}. \tag{21}$$

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we obtain a fractional definite integral formula. On the other hand, we provide some examples to illustrate the applications of this formula. In fact, our formula is a generalization of classical calculus formula. In the future, we will continue to use Jumarie's modified R-L fractional calculus and the new multiplication of fractional analytic functions to solve problems in fractional differential equations and applied mathematics.

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